

# Binary signals: a note on the prime period of a point

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## Abstract

The 'nice'  $x : \mathbf{R} \rightarrow \{0, 1\}^n$  functions from the asynchronous systems theory are called signals. The periodicity of a point of the orbit of the signal  $x$  is defined and we give a note on the existence of the prime period.

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The asynchronous systems are the models of the digital electrical circuits and the 'nice' functions representing their inputs and states are called signals. Such systems are generated by Boolean functions that iterate like the dynamical systems, but the iterations happen on some coordinates only, not on all the coordinates (unlike the dynamical systems). In order to study their periodicity, we need to study the periodicity of the (values of the) signals first. Our present aim is to define and to characterize the periodicity and the prime period of a point of the orbit of a signal.

**Definition 1** *The set  $\mathbf{B} = \{0, 1\}$  is a field relative to  $'\oplus'$ ,  $'\cdot'$ , the modulo 2 sum and the product. A linear space structure is induced on  $\mathbf{B}^n$ ,  $n \geq 1$ .*

**Notation 2**  $\chi_A : \mathbf{R} \rightarrow \mathbf{B}$  is the notation of the characteristic function of the set  $A \subset \mathbf{R} : \forall t \in \mathbf{R}$ ,

$$\chi_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{otherwise} \end{cases}.$$

**Definition 3** *The **continuous time signals** are the functions  $x : \mathbf{R} \rightarrow \mathbf{B}^n$  of the form  $\forall t \in \mathbf{R}$ ,*

$$x(t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \quad (1)$$

where  $\mu \in \mathbf{B}^n$  and  $t_k \in \mathbf{R}, k \in \mathbf{N}$  is strictly increasing and unbounded from above. Their set is denoted by  $S^{(n)}$ .  $\mu$  is usually denoted by  $x(-\infty + 0)$  and is called the **initial value** of  $x$ .

**Definition 4** The **left limit**  $x(t-0)$  of  $x(t)$  from (1) is by definition the function  $\forall t \in \mathbf{R}$ ,

$$x(t-0) = \mu \cdot \chi_{(-\infty, t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0, t_1]}(t) \oplus \dots \oplus x(t_k) \cdot \chi_{(t_k, t_{k+1}]}(t) \oplus \dots \quad (2)$$

**Remark 5** The definition of  $x(t-0)$  does not depend on the choice of  $(t_k)$  that is not unique in (1); for any  $t' \in \mathbf{R}$ , the existence of  $x(t'-0)$  is used in applications under the form  $\exists \varepsilon > 0, \forall \xi \in (t' - \varepsilon, t'), x(\xi) = x(t' - 0)$ .

**Definition 6** The set  $Or(x) = \{x(t) | t \in \mathbf{R}\}$  is called the **orbit** of  $x$ .

**Notation 7** For  $x \in S^{(n)}$  and  $\mu \in Or(x)$ , we denote

$$\mathbf{T}_\mu^x = \{t | t \in \mathbf{R}, x(t) = \mu\}. \quad (3)$$

**Definition 8** The point  $\mu \in Or(x)$  is called a **periodic point** of  $x \in S^{(n)}$  or of  $Or(x)$  if  $T > 0, t' \in \mathbf{R}$  exist such that

$$(-\infty, t'] \subset \mathbf{T}_{x(-\infty+0)}^x, \quad (4)$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x. \quad (5)$$

In this case  $T$  is called the **period** of  $\mu$  and the least  $T$  like above is called the **prime period** of  $\mu$ .

**Theorem 9** Let  $x \in S^{(n)}$ ,  $\mu = x(-\infty + 0)$ ,  $T > 0$  and the points  $t_0, t_1 \in \mathbf{R}$  having the property that

$$t_0 < t_1 < t_0 + T, \quad (6)$$

$$(-\infty, t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup [t_1 + 2T, t_0 + 3T) \cup \dots = \mathbf{T}_\mu^x \quad (7)$$

hold.

a) For any  $t' \in [t_1 - T, t_0)$ , the properties (4), (5) are fulfilled and for any  $t' \notin [t_1 - T, t_0)$ , at least one of the properties (4), (5) is false.

b) For any  $T' > 0, t'' \in \mathbf{R}$  such that

$$(-\infty, t''] \subset \mathbf{T}_{x(-\infty+0)}^x, \quad (8)$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t'', \infty), \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty) \subset \mathbf{T}_\mu^x, \quad (9)$$

we have  $T' \geq T$  and  $t'' \in [t_1 - T', t_0)$ .

**Proof.** a) Let  $t' \in [t_1 - T, t_0)$ . From

$$(-\infty, t'] \subset (-\infty, t_0) \subset \mathbf{T}_\mu^x,$$

we infer the truth of (4).

Furthermore, we have

$$\mathbf{T}_\mu^x \cap [t', \infty) = [t', t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup [t_1 + 2T, t_0 + 3T) \cup \dots$$

and we take an arbitrary  $t \in \mathbf{T}_\mu^x \cap [t', \infty)$ . If  $t \in [t', t_0)$ , then

$$\begin{aligned} \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) &= \{t, t + T, t + 2T, \dots\} \subset \\ &\subset [t', t_0) \cup [t' + T, t_0 + T) \cup [t' + 2T, t_0 + 2T) \cup \dots \subset \\ &\subset [t', t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup \dots \subset \mathbf{T}_\mu^x. \end{aligned}$$

If  $\exists k_1 \geq 0, t \in [t_1 + k_1T, t_0 + (k_1 + 1)T)$ , then there are two possibilities:

Case  $t \in [t' + (k_1 + 1)T, t_0 + (k_1 + 1)T)$ , when

$$\begin{aligned} \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) &= \{t + (-k_1 - 1)T, t + (-k_1)T, t + (-k_1 + 1)T, \dots\} \subset \\ &\subset [t', t_0) \cup [t' + T, t_0 + T) \cup [t' + 2T, t_0 + 2T) \cup \dots \subset \\ &\subset [t', t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup \dots \subset \mathbf{T}_\mu^x; \end{aligned}$$

Case  $t \in [t_1 + k_1T, t' + (k_1 + 1)T)$ , when

$$\begin{aligned} \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) &= \{t + (-k_1)T, t + (-k_1 + 1)T, t + (-k_1 + 2)T, \dots\} \subset \\ &\subset [t_1, t' + T) \cup [t_1 + T, t' + 2T) \cup [t_1 + 2T, t' + 3T) \cup \dots \subset \\ &\subset [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup [t_1 + 2T, t_0 + 3T) \cup \dots \subset \mathbf{T}_\mu^x. \end{aligned}$$

We suppose now that  $t' \notin [t_1 - T, t_0)$ . If  $t' < t_1 - T$ , we notice that  $\max\{t', t_0 - T\} < t_1 - T$  and that for any  $t \in [\max\{t', t_0 - T\}, t_1 - T)$ , we have  $t \in \mathbf{T}_\mu^x \cap [t', \infty)$  but

$$t + T \in \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \cap [t_0, t_1),$$

thus  $t + T \notin \mathbf{T}_\mu^x$  and (5) is false. On the other hand if  $t' \geq t_0$ , then  $x(t_0) \neq \mu$  implies  $t_0 \notin \mathbf{T}_\mu^x$  and consequently (4) is false.

b) The fact that  $t'' \in [t_1 - T', t_0)$  is proved similarly with the statement  $t' \in [t_1 - T, t_0)$  from a):  $t'' \geq t_0$  is in contradiction with (8) and  $t'' < t_1 - T'$  is in contradiction with (9).

We suppose now against all reason that (8), (9) are true and  $T' < T$ . Let us note in the beginning that

$$\max\{t_1, t_0 + T - T'\} < \min\{t_0 + T, t_1 + T - T'\}$$

is true, since all of  $t_1 < t_0 + T$ ,  $t_1 < t_1 + T - T'$ ,  $t_0 + T - T' < t_0 + T$ ,  $t_0 + T - T' < t_1 + T - T'$  are true. We infer that any  $t \in [\max\{t_1, t_0 + T - T'\}, \min\{t_0 + T, t_1 + T - T'\})$  fulfills  $t \in [t_1, t_0 + T) \subset \mathbf{T}_\mu^x \cap [t'', \infty)$  and

$$t_0 + T \leq \max\{t_1 + T', t_0 + T\} \leq t + T' < \min\{t_0 + T + T', t_1 + T\} \leq t_1 + T$$

in other words  $t + T' \in \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty)$ , but  $t + T' \in [t_0 + T, t_1 + T)$ , thus  $t + T' \notin \mathbf{T}_\mu^x$ , contradiction with (9). We conclude that  $T' \geq T$ . ■

**Lemma 10** *We suppose that the point  $\mu \in Or(x)$  is periodic:  $T > 0, t' \in \mathbf{R}$  exist such that (4), (5) hold. If for  $t_1 < t_2$  we have  $[t_1, t_2) \subset \mathbf{T}_\mu^x \cap [t', \infty)$ , then  $\forall k \geq 1, [t_1 + kT, t_2 + kT) \subset \mathbf{T}_\mu^x$ .*

**Proof.** Let  $k \geq 1$  and  $t \in [t_1 + kT, t_2 + kT)$  be arbitrary. As  $t - kT \in [t_1, t_2)$  and from the hypothesis  $t - kT \in \mathbf{T}_\mu^x \cap [t', \infty)$ , we have from (5) that

$$t \in \{t - kT + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x.$$

■

**Theorem 11** *We ask that  $x$  is not constant and let the point  $\mu = x(-\infty + 0)$  be given, as well as  $T > 0, t' \in \mathbf{R}$  such that (4), (5) hold. We define  $t_0, t_1 \in \mathbf{R}$  by the requests*

$$\forall t < t_0, x(t) = \mu, \quad (10)$$

$$x(t_0) \neq \mu, \quad (11)$$

$$t_1 < t_0 + T, \quad (12)$$

$$\forall t \in [t_1, t_0 + T), x(t) = x(t_0 + T - 0), \quad (13)$$

$$x(t_1 - 0) \neq x(t_1). \quad (14)$$

*Then the following statements are true:*

$$t_1 - T \leq t' < t_0 < t_1, \quad (15)$$

$$(-\infty, t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup [t_1 + 2T, t_0 + 3T) \cup \dots \subset \mathbf{T}_\mu^x. \quad (16)$$

**Proof.** The fact that  $x$  is not constant assures the existence of  $t_0$  as defined by (10), (11). On the other hand  $t_1$  as defined by (12), (13), (14) exists itself, since if, against all reason, we would have

$$\forall t < t_0 + T, x(t) = x(t_0 + T - 0), \quad (17)$$

then (10), (11), (17) would be contradictory. By the comparison between (10), (11), (13), (14) we infer  $t_0 \leq t_1$ . From (4), (10), (11) we get  $t' < t_0$ .

Case  $t' \in [t_1 - T, t_0)$

In this situation

$$t' + T \in [t_1, t_0 + T), \quad (18)$$

$$\mu \stackrel{(4)}{=} x(t') \stackrel{(5)}{=} x(t' + T) \stackrel{(13),(18)}{=} x(t_1) \quad (19)$$

and from (10), (11),  $t_0 \leq t_1$ , (19) we have that  $t_0 < t_1$ . (15) is true.

Let us note that (13),  $t' < t_1$ , (19) imply  $[t_1, t_0 + T) \subset \mathbf{T}_\mu^x \cap [t', \infty)$  and, from Lemma 10 together with (10), (11) we obtain the truth of (16).

Case  $t' < t_1 - T$

As  $t_1 - T < t_0$  we can write

$$\mu \stackrel{(10)}{=} x(t_1 - T) \stackrel{(5)}{=} x(t_1) \quad (20)$$

and the property of existence of the left limit of  $x$  in  $t_1$  shows the existence of  $\varepsilon > 0$  with

$$\forall t \in (t_1 - \varepsilon, t_1), x(t) = x(t_1 - 0). \quad (21)$$

We take  $\varepsilon' \in (0, \min\{t_1 - T - t', \varepsilon\})$ , thus for any  $t \in (t_1 - T - \varepsilon', t_1 - T)$  we have

$$t + T \in (t_1 - \varepsilon', t_1) \subset (t_1 - \varepsilon, t_1), \quad (22)$$

and since

$$t > t_1 - T - \varepsilon' > t', \quad (23)$$

$$t < t_1 - T < t_0 \quad (24)$$

we conclude

$$\mu \stackrel{(10),(24)}{=} x(t) \stackrel{(5),(23)}{=} x(t + T) \stackrel{(21),(22)}{=} x(t_1 - 0). \quad (25)$$

Equations (14), (20), (25) are contradictory, thus  $t' < t_1 - T$  is impossible. ■

**Example 12** We take  $x \in S^{(1)}$ ,

$$x(t) = \chi_{(-\infty, 0)}(t) \oplus \chi_{[1, 2)} \oplus \chi_{[3, 5)} \oplus \chi_{[6, 7)} \oplus \chi_{[8, 10)} \oplus \chi_{[11, 12)} \oplus \dots$$

In this example  $\mu = 1, t_0 = 0, t_1 = 3, T = 5$  is prime period and  $t' \in [-2, 0)$ . We note that  $T$  may be prime period without equality at (16) but if we have equality at (16) then, from Theorem 9,  $T$  is prime period.

**Remark 13** Theorems 9 and 11 refer to the case when the periodic point  $\mu$  coincides with  $x(-\infty + 0)$ . The situation when  $\mu \neq x(-\infty + 0)$  is not different in principle.

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